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Q
An Algebraic Language for Quantum-Spacetime Topology

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We propose a third physical logics. The first was classical (C) logics with commutative distributive AND and OR. The second is commutative quantum (CQ) logics, with commutative nondistributive AND and OR. The third logics, Q, has noncommutative nondistributive AND and OR; Q ⊃ CQ ⊃ C. Q predicates are the rays in a Grassmann double algebra of extensors, where CQ predicates are the subspaces of a Hilbert space. The AND and OR of Q are projectively represented by Grassmann's progressive and regressive products. Q supports a quantum set theory appropriate to quantum topology. Here Q is applied to a toy theory of the topology of time. It preserves translational invariance and replaces singular delta-function propagators by finite Gaussians.

1. INTRODUCTION

In classical theoretical physics we ultimately express all variables of any stem under study, and relations among them, in the universal language of set theory. Von Neumann (1932) recognized that quantum theory modified the underlying logic and set theory for quanta, and spoke of a "quantum set theory" that would presumably play the same universal role for quantum physics that the usual, or classical, set theory does for classical physics; but he did not complete its construction. Von Neumann (1932) and Birkhoff and von Neumann (1936) proposed that the logical particles of quantum logics, the quantum equivalents of AND, OR, NOT, and IF, do not obey the laws of Boolean algebra, but those of the orthomodular lattice of closed linear subspaces of a Hilbert space. Predicates, classes, and sets of quanta are represented by Hilbert subspaces. These AND and OR operations are idempotent, commutative, and nondistributive.

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This commutative quantum (CQ) logics has a strong classical component. The predicates it represents by vectors of a Hilbert space are of order 1 (predicates about quanta, not about predicates) and degree 1 (pure cases, not mixtures). These are subject to quantum superposition. Its higher-order and higher-degree predicates are not. This is physically reasonable for macroscopic operations, but it is also limiting, in that it excludes microscopic quantum operations. In this paper we propose as a further step toward a universal algebraic quantum physical language a noncommutative one called Q which is richer than CQ in much the way that CQ is richer than C, having a still stronger superposition principle; namely, one which applies to higher-degree and higher-order predicates as well as to first-degree, first-order predicates. This goes beyond the Copenhagen quantum theory, which insists that the observer is classical.

In its simplest and most natural form Q forces us to regard all bosons as pseudobosons composed of fermions. The generalization to admit the existence of proper bosons is straightforward but not plausible and is omitted here.

Our search for Q began with the observation (Finkelstein *et al.*, 1959; Beth, 1961) that a mathematical set is actually a Fermi–Dirac ensemble, in that its occupation numbers are restricted to 0 and 1 and the interchange of its elements is ignorable. Further, a physical set of electrons is a Fermi–Dirac ensemble. We therefore constructed a quantum set theory by replacing the classical power-set operation

$$P: \quad X \mapsto P(X)$$

by the operation of Fermi–Dirac "second quantization"

$$Q: \quad H \mapsto Q(H)$$

transforming each Hilbert space H into the normed exterior algebra over H. This observation led successively to a spacetime code, to Clifford and Grassmann algebraic candidates for Q, and to the recent proposal (Finkelstein, 1987) that the quantum correspondent of $P(X)$ is more accurately the functor $E(H)$, the extensor algebra over H defined below. In all that work the Hilbert-space-based quantum logics itself was assumed to be correct, until now.

Now we note that even the first-order quantum predicate algebra of the usual quantum theory must be revised in the process of constructing this quantum set theory. The arena E for the higher-order quantum logics is not merely a Hilbert space but also an extensor algebra. For uniformity this must also be true for the first-order logics. This has the following consequences.

The normed extensor algebra $E(H)$ admits operations corresponding to AND and OR, which we call QAND and QOR, which are projectively represented, as described below, by Grassmann's progressive and regressive products of extensors. These Q operations are still further from Boolean algebra than the CQ operations, being noncommutative as well as nondistributive. Nevertheless, they reduce to familiar commutative orthomodular lattice operations in appropriate special cases. Q is not a lattice, but a "noncommutative lattice"; and not a poset but a "noncommutative poset".

The system for which Q is designed is quantum-spacetime, a synthesis of the theories of quanta and spacetime. From the beginning of quantum theory it was recognized that a quantum theory of gravity was needed to take into account the reaction of quanta upon the gravitational fields that guide them. Latter-day developments in gravity theory and cosmology call for such a synthesis in order to describe cosmogenesis and other events which appear as singularities in the continuum theory.

This synthesis seems to require a quantum-spacetime topology, on the following grounds:

Since the resultant of many future-null connections is timelike or null but not spacelike, the principle of causality can be reduced to the principle that only future-null connections actually exist in the microtopology of nature. But what is future-null is determined by gravity and is itself a quantum variable. Therefore a quantum theory of gravity likely waits upon a quantum theory of microtopology.

This leads one to entertain cellular alternatives to the continuum, along the lines of cellular automata or neural networks whose sole dynamical variable is the connection pattern, the physical topology.

A topology is at least a set of sets, namely the open sets, and so the first attempt at a quantum topology ought to be a quantum set of quantum sets. Events are assigned order 0, sets of events order 1, and sets of such sets are said to be of order 2. (We apply the term order uniformly to both predicates and sets.) Thus, a quantum topology calls for a quantum theory of higher-order sets, or equivalently, of higher-order predicates. Other ways of expressing topology have been considered besides the open sets, but they all use higher-order set theory.

CQ provides a quantum set theory only of the lowest order and degree. Second-order predicates (predicates about predicates) are Boolean in the CQ logics. Q, however, includes a higher-order quantum logics. Here we develop the language Q and apply it to a toy model of quantum time.

2. EXTENSOR ALGEBRA

Birkhoff and von Neumann (1936) arrived at lattice logics by abstraction from the subspaces of Hilbert space, which form the predicates of the empirical logic of quantum mechanics. CQ logics is an algebraic theory of the subspaces of Hilbert space. CQ logics is algebraic only in the generalized sense of universal algebra, not in the sense of linear algebra, for it has no additive group.

Grassmann (1844) already proposed a remarkable algebraic theory of subspaces, and one quite different from von Neumann's, being linear algebraic. It is obviously superior to CQ lattice algebra for practical quantum physics, for it includes the algebra that is customary for fermionic quanta. Grassmann's theory, however, is not what has become known as Grassmann algebra, which omits half of Grassmann's structure. The alternative which Grassmann provides to the lattice $L(H)$ of H is called the extensor algebra over H, designated by $E(H)$. Q logics use Grassmann's theory, suitably extended, rather than von Neumann's.

We begin with a summary of the CQ logics to facilitate the transition to Q. CQ logics start from a Hilbert space H of dimension N (possibly $N=\aleph_0$). As a Hilbert space, H is provided with a Hilbert anti-isomorphism $\dagger$ to its dual space $\dagger H$; we write all operators to the left of their operands, even the operators $\dagger$ and $*$.

Instant-based (synchronic, prerelativistic) quantum theory deals with two basic modes of operations on a physical system, called input and output, represented in the simplest case by vectors of H and dual vectors of $\dagger H$, respectively. (This description of quantum theory comes closer both to current physics practice and to the Copenhagen form of quantum theory than does von Neumann's, who spoke of only one mode of input/output operation, "measurement," represented by projectors. The difference between the bimodal and the unimodal formulation is particularly significant for quantum cosmology and thus for quantum-spacetime. We adopt the bimodal logics both for Q and CQ.)

The diachronic (a.k.a. functional, relativistic) quantum theory describes input and output operations as positive and negative frequency subspaces of one space of "sources," but is still bimodal in that it reserves the dual space for "fields" (in the generalized sense that includes action principles). For the sake of familiarity we shall interpret Q synchronically at first, and diachronically only later, in the relativistic applications of Q.

First we summarize the properties and notation of the extensor algebra $E=E(H)$ over a given Hilbert space H; then we discard H in favor of E. H is only scaffolding. Q is based directly upon an extensor algebra E, and H itself survives in Q only as one of two completely symmetric subspaces of E. In the principle application we construct E inductively from $\mathbb{C}$ alone.

The elements of structure of extensor algebra are schematically

$$\langle\alpha|, +, \mathbb{C}, |\beta\langle, |\beta\langle\alpha|, \vee, \wedge, *, \dagger$$

which we take up in turn.

Extensors. The elements of an extensor algebra $E(H)$ are called *extensors* (*lineale Ausdehnungen, extensive Grössen* by Grassmann) over *H*. These are the simpler nontrivial words of the language Q. Like Dirac's kets, they represent coherent uniform quantum input operations transferring a quantum from experimenter to experimentee, the beginnings of experiments (in the synchronic theory; later, sources). These we call input extensors and write as $\langle\alpha|$ or Ψ or (Ψ^{α}). Extensors form a complex linear space, with addition + and multiplication by elements of $\mathbb{C}$ having the usual quantum meanings. (We anticipate reducing the complex coefficients to integers in a later, more fundamental theory, based upon an extensor module rather than an extensor algebra.)

Dual Extensors. Dual extensors are linear functions $E \to \mathbb{C}$ and make up a space $\dagger E(H)$. Like Dirac's bras, they represent outputs, the ends of experiments (later, fields). They are written $|\beta\langle$ or Φ or (Φ_n) so that an operator can be written as an arrow $\langle\beta|\alpha\langle$, representing throughput. The value or contraction of a dual extensor ϕ with an extensor ψ is written $|\beta\langle\alpha|$ or $\Phi(\Psi)$ or $\Phi_n\Psi^n$; its physical meaning is that it vanishes for forbidden transitions.

Hilbert dual. $E(H)$ is a Hilbert space like *H*. For every extensor ψ there is given a Hilbert-dual extensor $\dagger\psi$ associated with ψ by the Hilbert dual or adjoint operation $\dagger$, an (antilinear) anti-isomorphism $\dagger: E \to \dagger E$ induced by the usual adjoint $\dagger$ already given on $H \subset E(H)$. The Hilbert norm of an extensor Ψ is written $\|\Psi\|$ or $\dagger\Psi\Psi$.

Two Products. Like a lattice, an extensor algebra $E(H)$ has two associative operations, designated by $\vee$ and $\wedge$ (QOR and QAND, quantum noncommutative forms of OR and AND), with respective units $\downarrow$ and $\uparrow$; Grassmann's progressive and regressive products of extensors. The classical correspondent of the $\vee$-product is a partial operation different from OR and XOR which was introduced by C. S. Peirce and has long been known as the disjoint union; we designate it by POR and its dual by PAND.

Degree. The extensor algebra $E(H)$ has a grade g called "degree" (Grassmann's *Stufe*) ranging from 0 to *N*, the dimension of *H*; and another grade $N-g$, the codegree. Degrees add for $\vee$-products, and codegrees add for $\wedge$-products. *H* is the first-degree part of $E(H)$. If *N* is infinite (Grassmann did not develop this case), *E* is a direct sum of a subspace of finite degree and infinite codegree and a subspace of infinite degree and finite codegree. This differs from the lattice logic of *H*, which includes projections whose dimension and codimension are both infinite. The degree g corresponds to the modulus of lattice theory and the multiplicity (degree of degeneracy) of spectroscopy.

Hodge dual. Grassmann placed the greatest possible emphasis upon a dual symmetry between $\wedge$- and $\vee$-products, expressed by an antilinear mapping

$$*: \quad E(H) \to E(H)$$

called the Hodge dual nowadays. Grassmann even refused to introduce distinct multiplication signs for the two products, rather than disturb this perfect symmetry; one had to infer which products he meant in a formula from context. We use the signs $\wedge$ and $\vee$ of Peano, but in every other way we maintain Grassmann's symmetry. The Hodge dual * is QNOT, the Q NOT, and complements degree; if ψ has degree g, then $*\psi$ has degree $N-g$. If N is infinite and g is finite, $N-g$ is understood to be infinite.

Units. E is an exterior algebra in two ways: with exterior product $\vee$ and unit $1=\downarrow$, and with exterior product $\wedge$ and unit $*1=\uparrow$. E is closed under finite $\vee$-products and $\wedge$-products. This statement includes null $\vee$-products and $\wedge$-products; and the $\vee$-product of no elements is $1=\downarrow$, while the $\wedge$-product of no elements is $*1=\uparrow$. Thus, this statement implies that both units belong to E.

The Hilbert and Hodge operators † and * uniquely determine each other. They are the same element of structure expressed in different terms. If ψ is an input extensor, or creator, the Hilbert dual $*\psi$ outputs (or annihilates) the same kind of quantum that ψ inputs; while the Hodge dual $\dagger\psi$ inputs every other kind of quantum but that.

The most general extensor of E is obtained from the vectors of H by a finite number of $\vee$-products, and linear combination, or is the Cauchy completion of such with respect to the norm $\|\psi\| = \dagger\psi\psi$; and dually for the construction of E from $*H$.

There are several works on extensor algebra that can be consulted for a fuller picture, especially Barnabei *et al.* (1985) and works cited there.

3. EXTENSOR LOGICS

C, CQ, and Q logics correspond to extensors of three ascending levels of generality relative to a special basis $b \subset H$ called classical:

1. Extensors which are $\vee$-products of basis vectors of b are called classical; they support a classical logic (commutative distributive).

2. Extensors which are $\vee$-products of arbitrary vectors are called simple. They support a CQ quantum logic (commutative nondistributive).

3. Extensors which are not simple are called compound. The general extensors, simple and compound together, support the quantum logic of Q (noncommutative, nondistributive).

To say this more fully, we require the following concepts.

We designate the ray of an extensor Ψ by $[\Psi]$. If $[\Psi]=[X]$, we write $\Psi \equiv X$ and say that Ψ is projectively equal to X. We call $[\Psi]$ basic, simple, or compound according as Ψ is. We call a subspace of H basic that is a span of basic vectors.

In the logical interpretation of extensors there is a certain precise analogy between the extensor operations $\vee$, $\wedge$, $*$ and the respective logical operations $\cup$, $\cap$, $\neg$. To put all three logics in uniform terms we define partial operations for lattices akin to the extensor operations $\vee$ and $\wedge$, and express the lattice operations $\cup$ and $\cap$ in terms of them. We need not do this for an arbitrary lattice, but only for physical ones, which we suppose to be atomic ortholattices.

Let L be an atomic ortholattice with OR and AND operations $\cup$ and $\cap$, and write "0" for "undefined."

We may use OR ($\cup$) and AND ($\cap$) to define classical partial operations POR ($\vee$) and PAND ($\wedge$) by

$$\begin{aligned} \alpha \vee \beta &:= \alpha \cup \beta && \text{if} \quad \alpha \cap \beta = \downarrow \\ &= 0 && \text{otherwise} \\ \alpha \wedge \beta &:= \alpha \cap \beta && \text{if} \quad \alpha \cup \beta = \uparrow \\ &= 0 && \text{otherwise} \end{aligned} \tag{1}$$

From the viewpoint of classical logic, POR and PAND are pathological as well as partial. In particular, the truth value of α POR β is not a function of the truth values of α and β. Nevertheless, the classical operations of PAND and POR are better starting points for the passage to quantum theory than the Boolean operations; we shall take $\vee$ and $\wedge$ as fundamental and $\cup$ and $\cap$ as derived.

Proposition. For an atomic lattice, the lattice operations $\cup$ and $\cap$ and the partial operations $\vee$ and $\wedge$ defined by (1) uniquely determine each other.

Proof. How $\cup$ and $\cap$ determine $\vee$ and $\wedge$ is stated by the definitions (1). For the converse determination, first one constructs the lattice partial order $\alpha \leq \beta$ from $\vee$. For an atomic lattice, the lattice partial order relation $\alpha \leq \beta$ is equivalent to the assertion that for lattice atoms ψ, $\alpha \vee \psi = 0$ implies $\beta \vee \psi = 0$. And we may recognize the lattice atoms in terms of $\vee$ and $\wedge$ by their primeness: they cannot be written as $\vee$-products except trivially. Thus, $\vee$ and $\wedge$ determine $\leq$. Then the $\leq$-relation determines $\cup$ and $\cap$ as supremum and infimum in the usual way. ■

Proposition. Q logics properly includes C and CQ logics as special cases as follows:

1. C case: For basic rays in $E(H)$, the extensor operations $\vee$, $\wedge$, and $*$ are isomorphic to the classical operations $\vee$, $\wedge$, and $\neg$ defined by (1) for the Boolean algebra of the subsets of the basis b. In particular, basic extensors ψ, χ, ϕ commute projectively:

$$\psi \vee \chi \equiv \chi \vee \psi$$

$$\psi \wedge \chi \equiv \chi \wedge \psi$$

and partially distribute projectively, in the sense that the distributive laws

$$\psi \wedge (\chi \vee \phi) \equiv (\psi \wedge \chi) \vee (\psi \wedge \phi)$$

$$\psi \vee (\chi \wedge \phi) \equiv (\psi \vee \chi) \wedge (\psi \vee \phi)$$

hold when both sides are defined.

2. CQ case: For disjoint simple rays the same operations agree with those of an orthomodular complemented lattice, the usual Hilbert space lattice of subspaces of H. In particular, simple extensors commute projectively: $\psi \vee \chi \equiv \chi \vee \psi$, but neither of the operations $\vee$ and $\wedge$ restricted to simple extensors distributes over the other.

3. Q: For disjoint general rays these operations do not agree with those of a lattice, since compound extensors do not commute projectively.

Proof. Straightforward verification. ■

Similar considerations may be carried out for the (partial) order $\leq$. Due to noncommutativity, an extensor algebra E defines two (and more) inequivalent orders, which we write as $\alpha \leq \beta$ and $\beta \geq \alpha$ and define by

$$\begin{aligned} \alpha \leq \beta &:\Leftrightarrow: (\forall \psi)(\psi \vee \alpha = 0 \Rightarrow \psi \vee \beta = 0) \\ \beta \geq \alpha &:\Leftrightarrow: (\forall \psi)(\beta \vee \psi = 0 \Leftarrow \alpha \vee \psi = 0) \end{aligned} \tag{2}$$

These reduce to a single Boolean order, a single lattice order, and distinct (and we expect nonlattice) orders in C, CQ, and Q logics, respectively.

Thus, we see that Q logics is a proper noncommutative generalization of the commutative CQ logics.

4. PHYSICAL INTERPRETATION

To ground this extension of quantum logics in experiment, it is helpful to recall a certain systematic difference between the way Heisenberg and Bohr interpret quantum (io) vectors and the way that Born and von Neumann do.

For background, we first recall a similar difference between two concepts of a heat reservoir R in equilibrium with a system S, say a gas, in statistical thermodynamics:

1. We can think of R as an individual, much larger body in thermal equilibrium with the individual system S.

2. We can represent R as a large ensemble of systems identical to S.

We may call these the individual and the statistical representation of R. The two interpretations are expected to be physically equivalent. Schrödinger (1938) adopts the statistical representation of the heat reservoir because it is more definite and simpler.

Similarly, we can think of an input vector ψ in two physically, equivalent ways, individual (like Bohr and Heisenberg) or statistical (like Born and von Neumann): Either:

1. The vector ψ represents an arbitrary way of preparing the individual quantum system S; a preparation in the sense of Ludwig, or a member of a manual in the sense of Foulis and Randal, for example. Briefly, ψ describes the entire experimental operation.

2. Or the vector ψ represents a more specific way of preparing the individual quantum system S, namely by random selection from an ensemble of quantum systems isomorphic in structure to S. Briefly, ψ describes the operation of selection from a set of S's.

The statistical interpretation is thus a special case of the individual one, in which the input device is a collection of quantum systems similar to the one being prepared. In either case the vector ψ describes not only the experimentee, but even more the experimenter.

In quantum theory as in thermodynamics, the statistical representation of an input operation is simpler and more definite, and we adopt it. It must not be supposed, however, that ψ represents an ensemble of points in a classical phase space, say; it represents an ensemble of physical systems admitting no phase space description, not an ensemble of classical systems (since such do not exist in nature).

Pure, Crisp, Coherent, and Statistical Ensembles. Noncommutative quantum logics arise because there are more physical sets of quanta than are envisaged in the CQ lattice logics. Besides those described by subspaces, the elements of the usual CQ lattice, or the simple extensors of the Q logics, there are those described by compound extensors, coherent superpositions of simple extensors. Every subspace is represented by an extensor, with arbitrary phase, but almost no extensors represent a subspace.

To avoid confusion it is helpful to distinguish four degrees of specification of a quantum, each a special case of the next: sharp, crisp, coherent, and statistical (or fuzzy).

1. In C logics with discrete possibility space S, a sharp description is a point $p \in S$, a crisp one is a subset $P \subset S$, and a statistical one is a probability distribution ρ on S.

2. In CQ logics, the sharp (pure, atomic, maximal) description is a vector $\psi \in H$, the crisp description is a projection operator P in H, the coherent description is restricted to the sharp case, and the most general statistical description is by a statistical operator ρ on H, assigning probabilities to pure ensembles.

3. In Q logics, the sharp description is a first-degree extensor, the crisp is a simple extensor, the coherent is an arbitrary extensor, and the statistical is a statistical operator on $E(H)$, assigning probabilities to coherent ensembles. The CQ logics deals with crisp ensembles, and the Q logics with coherent.

We limit ourselves here to a Fermi–Dirac system F. (The analogous theory for bosons lacks the Hodge duality and the $\wedge$-product.) By a set of F's we mean exactly a Fermi–Dirac ensemble of F's, on the grounds that for both sets and Fermi–Dirac ensembles, the occupation numbers are restricted to 0 and 1 and the order of systems is ignorable. Since Fermi–Dirac ensembles are maximally described by multivectors (skew tensors) over H, we may specify a set of F's by a multivector over H. In the presence of the Hilbert inner product, a multivector is also an extensor.

It is easy to imbed the CQ lattice logics within the Q extensor logics. Each element of the Q logics is a subspace P, and may be represented as Grassmann intended by the simple extensor formed by $\vee$-multiplying the vectors in a basis for P.

In addition there are compound extensors. Grassmann had no meaning for these precisely because they do not represent subspaces. They represent quantum superpositions of preparations described by subspaces, and represent physically possible preparations not represented in QC logics.

For example, extensors of degree 1 represent "pure states" in the CQ logics, while those of degree 2 represent "mixed states," mixtures of two degree-1 inputs. A coherent superposition of a degree-1 and a degree-2 input, or of two grade-2 inputs, does not occur in the CQ quantum logics, but does in Q.

An interpretation of an input vector is an experimental input operation described by the vector. It is not difficult to design physical input operations to go with these theoretical ones. We may carry out such an operation in two steps: the formation of a coherent ensemble, and the extraction of a member of that ensemble:

Formation. A helium atom with its ground energy is a coherent ensemble of two electrons. To form a coherent superposition of a one-fermion and two-fermion input, to be sure, violates the statistics superselection rule of Wick, Wightman, and Wigner, and requires one to take into account quantum variables of the experimenter which are usually not considered. But no such problem arises in the formation of coherent superpositions of two degree-2 input vectors; for example, a difermion of spin $S = 1$ may be resolved by a Stern–Gerlach operation into a coherent superposition of vectors with z component of spin $S_z = 1$, 0, and -1.

Extraction. The operation of extracting one member of a coherent ensemble at random may be approximated by a stripping reaction, where a projectile combines with one particle of a target and carries it off. We must carry out the stripping operation on the contents of the target region without determining the contents more precisely than by the prior preparation.

The duals to these input operations are output operations. We omit their physical description here. To retain all phase data it may be necessary to use what is left of the input ensemble after the extraction operation—in the given example, an He^+ ion—in carrying out the output operation.

The inclusion of such a quantum ensemble (say, a single helium atom with its minimum energy) as part of the apparatus is not envisaged in Copenhagen quantum theory. To be sure, the CQ logics could describe this input operation by a statistical operator in the single-fermion Hilbert space. This would lose phase information that is retained in the extensor description. The CQ logics could also describe the new one-fermion experiments of Q as special many-fermion experiments. But the extensor logics of many fermions is richer still. At the end, for any given physical system the Q logics is a proper extension of the CQ quantum logics; it has a larger class of predicates.

We may think of an extensor Ψ as explicitly representing the creation of a Fermi–Dirac ensemble, to be followed implicitly by the selection of a member. This is the extensional way of defining a class: by giving its members. The product $\Psi \vee \Phi$ represents the successive execution of the two creation operations Φ and Ψ, taking into account the Pauli exclusion principle. QOR is noncommutative because in general the result of such a sequence of creation operations depends on the order of execution.

Q is thus considerably simpler than CQ. The CQ logics describe predicates and sets (that is, Fermi–Dirac ensembles) of electrons by quite different categories of algebras: lattices and exterior algebras, respectively. In C logics finite predicates and sets are isomorphic, the sets being the extensions of the predicates. Q restores this isomorphism.

5. HIGHER-ORDER QUANTUM LOGICS

So much for the first-order predicate algebra. The principle gain from this extension of the CQ Hilbert-space-based quantum logics is its higher-order quantum predicate algebra. This theory is still somewhat speculative, and is directed toward theories of quantum-spacetime that are not yet closely connected with experiment; but the simplicity and self-consistency of the higher-order Q theory is strong formal evidence for Q.

Classical Construction. The basic C operation for constructing higher-order predicates from first-order ones is the brace $\{\cdots\}$ of set theory. In general if ψ is a predicate, then $\{\psi\}$ is defined to be the predicate of predicates which holds only for the predicate ψ. That is, $\{\psi\}(x)$ means that $x=\psi$; $\{\psi\}$ is the property of being ψ. On the other hand, if ψ is interpreted as a set, then $\{\psi\}$ represents a set whose sole element is ψ. We may also think of the brace purely formally, as a way of making from any set of elements of arbitrary degree an isomorphic replica consisting of monads (first-degree sets). For instance, from a triad a, b, c we make a new triad $\{a\}, \{b\}, \{c\}$.

One might just as well use primes or bars for this purpose as braces, writing a', b', c' for the monadic replicas of a, b, c. Following Peano, however, we designate $\{\psi\}$ by $\iota\psi$; we used $Q\psi$ for $\iota\psi$ in Finkelstein, Jauch, and Speiser (1959) (because $Q\psi$ belongs to the quantified or second quantized theory) before we knew of Peano's work, which we learned about in Barnabei *et al.* (1985). We form finitely-constructible higher-order predicates over an object with first-order predicates C by applying to the elements of C the operations of $\vee$ and ι.

The Hodge dual of the empty set $\downarrow$ is a universe set $\uparrow=\neg\downarrow$ and of ι is an operator $\kappa=\neg\iota\neg$ [that is, for any set α, $\kappa\alpha:=\neg(\iota(\neg\alpha))$], producing sets of codegree 1. Thus, the null set $\downarrow$ breaks the negational symmetry between ψ and $\neg\psi$, the classical correspondent of the dual symmetry of Grassmann (Hodge dual). In physics this symmetry is broken by causality; points of spacetime are local and their complements are not. In set theory it is broken by the operator ι, since $\iota\psi$ has degree 1, and not codegree 1. We therefore use ι to express causal structure.

To avoid the paradox of $\iota(\uparrow)$, which cannot be an element of $\neg\downarrow$, we explicitly restrict the definition of ι to finite sets, defining $\iota\alpha=0$ for infinite sets α. Similarly, κ is defined only for cofinite sets.

For simplicity we now limit attention to pure set theory, with no proper elements. In the classical theory this means taking for the initial class algebra $C=1$ (the null set). In the quantum theory, it means taking for the initial Hilbert space $H=\mathbb{C}$ (the ray representing the null set). There is then only one lowest-order predicate, the null predicate, designated in extensor language by the number $1\in\mathbb{C}$. The corresponding set is the null set, and is conventionally assigned order 1.

In classical logics we apply ι to the first-order predicate 1 to construct a second-order predicate $\iota 1=\{1\}$; then $\{1\}$ represents the property of being the predicate 1. We describe an inductive process for making a Boolean algebra SET(C) of sets or predicates of all ranks from an arbitrary finite class C of first-order predicates or sets.

We define the single-element Boolean algebra $B^1 = \{1\}$ consisting of the null set.

Given the Boolean algebra B^N, the predicates of B^{N+1} are constructed by bracing the elements of B^N and closing with respect to the Boolean operations 1 and $\vee$. (One may equally say: with respect to arbitrary finite $\vee$-products. For the operator 1 is the $\vee$-product of no factors, the $\vee$-product of one factor is already present, and $\vee$ gives the $\vee$-product of two factors; the other finite $\vee$-products follow.) Thus, if B^N has cardinality $C(N)$, then the cardinality $C(N+1)$ of B^{N+1} is $C(N+1) = 2^{C(N)}$.

Proposition. $B^N \subset B^{N+1}$.

Proof. By induction on N. $N = 1$: Evidently $B^1 \subset B^2$. $N \to N+1$: If $B^N \subset B^{N+1}$, then (since the generating operations ι and $\vee$ preserve inclusion) it follows that $B^{N+1} \subset B^{N+2}$. ■

We now define SET, the set of semifinite sets. (SET is not itself a semifinite set.) For brevity, we use the term "finite set" for a set which has a finite number of elements, which also have a finite number of elements, which also have a finite number of elements, and so forth. A cofinite set shall be one whose complement is a finite set in this sense. And a semifinite set is a set that is either a finite set or a cofinite set.

Let B^∞ be the limit (that is, the union) of all the B^N. The set B^∞ is the set of all finite sets, which may be generated from $\downarrow$ by finite numbers of the operations $\vee$ and ι. While each B^N is closed under relative complementation $\alpha \mapsto \neg\alpha = B^N/\alpha$, the limit B^∞ is not. For each element α of B^∞ is finite by construction, but B^∞ is infinite, and the complement of α in B^∞ is therefore infinite and does not belong to any B^N, nor, it follows, to B^∞. Similarly, B^∞ has a null set $\downarrow$ but no universal set $\uparrow$. Our construction, based on $\vee$ alone, has spoiled the dual symmetry of Grassmann.

To repair this we shall (loosely speaking) reflect B^∞ in the "plane" of infinite sets. "Below" the plane are the finite sets; "above," the cofinite. There are no sets which are both; and those which are neither are omitted from SET.

More formally, we carry out a dual construction based on a formal universal class $\uparrow$, the $\wedge$-product $\wedge$, and κ, instead of $\downarrow$, $\vee$, and ι, leading now to a dual sequence of class algebras $^*B^N$ and their union $^*B^\infty$. Each set in $^*B^\infty$ is generated by a finite number of the operations $\wedge$ and κ, and is cofinite. The co-order of a set is the number of nested κ's in sequence in its construction from $\uparrow$ by $\wedge$-products and κ's. Finite co-order and codegree imply infinite order and degree.

SET is defined as the union $B^\infty \cup {}^*B^\infty$. Its elements are semifinite sets.

Proposition. SET is closed under $\vee$, $\wedge$, and $\neg$.

Proof. If $\psi \in$ SET, then either $\psi \in B^{\infty}$ or $\psi \in {}^*B^{\infty}$. In the former case, for some N, $\psi \in B^N$. Then $\neg\psi$ is defined as an element of $B^{*N} \subset {}^*B^{\infty} \subset$ SET. In the latter case we proceed dually. ■

The operations $\vee$, $\wedge$, and $\neg$ act in SET as shown in Table I.

Proposition. The operations $\vee$ and $\wedge$ on SET are commutative and distribute over each other.

Proof. By construction. ■

It is evident that SET is not closed under ι; in particular, $\iota(\uparrow)$ is undefined: $\iota(\uparrow)=0$. This is the ransom we pay for release from Cantor's paradox of the set of all sets. Moreover, while SET is closed under $\vee$ and $\wedge$, it is not closed under the Boolean operations $\cup$ and $\cap$; such closure would require a much larger class of sets.

Proposition. Each set $\alpha \in$ SET is either finite or cofinite (of finite complement in SET).

Table I. Operations on Finite and Cofinite Sets[a]

$\vee$	Φ	Φ^*
Φ	Φ	Φ^*
Φ^*	Φ^*	0

$\wedge$	Φ	Φ^*
Φ	0	Φ^*
Φ^*	Φ^*	Φ^*

	$\neg$
Φ	Φ^*
Φ^*	Φ

[a] Φ stands for a general finite set and Φ^* for a general cofinite set. For example, from the $\vee$-table read that the $\vee$-product of a finite set with a cofinite one is cofinite ("$\Phi \vee \Phi^* = \Phi^*$").

Proof. By induction. ■

If $\psi, \ldots, \chi$ are all distinct Nth-order predicates, then an example of an $(N+1)$th-order predicate is

$$\{\psi, \ldots, \chi\} = \iota\psi \vee \cdots \vee \iota\chi \tag{3}$$

Quantum Construction. The beauty of the extensor logics is that it provides these classical procedures with close parallels in the quantum theory. Heuristically speaking, we do to basis vectors of a vector space H what is classically done to points of a possibility space S. For example, the extensor algebra $E(H)$ over a vector space H is the quantum correspondent of the Boolean algebra $B(S)$ over an arbitrary set S. The quantum correspondent of the mapping ι is a semilinear transformation ι. In some applications it is most convenient to take ι to be antilinear; here, for simplicity of exposition, we assume ι linear.

To begin our inductive construction, we define $E^0 = \mathbb{C}$, the complex numbers. This simplest possible extensor algebra, in which $\vee$ and $\wedge$ coincide with ordinary complex multiplication, represents the null set (or the fermionic vacuum).

Given the extensors of order N, we form those of order $N+1$ by bracing all the elements of a basis of E^N (applying ι) and closing the resulting set under finite $\vee$-products and linear combination. The Hilbert dual $\dagger$ in E^{N+1} is defined in terms of matrix elements of ι and $\dagger\iota\dagger$, which we determine below.

By induction on N we form extensors of all ranks E^R; the least value of R for which E^R contains an extensor ψ is called the order of ψ. The union of the E^R is an infinite-dimensional exterior algebra E^∞ whose elements are called finite extensors. The basic extensors of E^∞ are generated by finite numbers of the operations $\vee$ and ι subject to familiar identities. Every basic element of E^∞ is of finite degree and order and represents a finite set.

We form a space $*E^\infty$ dual to E^∞ by the dual process, replacing $\downarrow$, $\vee$, and ι by $\uparrow$, $\wedge$, and κ. We define extensor operations $\vee$, $\wedge$, and $*$ on the direct sum $E^\infty \oplus *E^\infty$ in the way we have already rehearsed with $\text{SET} = B^\infty \cup *B^\infty$ (Table I).

In the quantum construction there is one topological step that has no counterpart in the Boolean one: We complete the extensor algebra $E^\infty \oplus *E^\infty$ to form the Hilbert space of quantum sets, which we call QET. There is a natural Hilbert norm on E^∞, on $*E^\infty$, and therefore on $E^\infty \oplus *E^\infty$. Cauchy-completion of $E^\infty \oplus *E^\infty$ with respect to that norm yields the Hilbert space QET. Extensors in QET may be called qets.

We suppose that all Fermi–Dirac objects in nature arise from the anticommutativity of first-degree qets.

We turn now to normalizing the operator ι. The operator ι raises order and its Hermitian adjoint

$$ {}^{\dagger}\iota := \dagger\iota\dagger \tag{4} $$

lowers order. It is natural to follow the precedent of the harmonic oscillator raising and lowering operators when we normalize ι. To be sure, E is not a ladder with ι as raising operator, because of its branching structure, which causes a hyperexponential growth in dimension with order. Nevertheless, the following harmonic oscillator result is valid here too.

By a classical qet we mean a nonzero qet that is constructed exclusively by finitely many operators $\downarrow$, $\vee$, and ι and their Hodge duals (without linear combination); the classical qets have rays also called classical, which are in an obvious isomorphism to the classical sets in SET. Classical qets are not normalized to unity and are either finite or cofinite. We have constructed QET so that the action of the quantum ι on the classical rays in QET is isomorphic to that of the classical ι on the sets of SET.

We also use the following concepts of graph and extent.

Each classical qet of finite degree has a finite graph, a family tree showing its evolution from the null set, with the vertices

—•< (two input lines) for each dyadic operator $\vee$
—•— (one input line) for each monadic operator ι
—• (zero input lines) for each cenadic operator $\downarrow$

We define the rank of a classical extensor or set of finite degree to be the number of ι vertices in sequence at the right end of its family tree.

Similarly, extensors of finite codegree have graphs showing their evolution from the universal set $\uparrow$, with the vertices

—o< (two input lines) for each dyadic operator $\wedge$
—o— (one input line) for each monadic operator κ
—o (zero input lines) for each cenadic operator $\uparrow$

Proposition. There exists a unique operator ι defined on a dense subspace in QET such that:

1. The operator ι obeys the relation

$$ {}^{\dagger}\iota\iota - \iota{}^{\dagger}\iota = 1 \tag{5} $$

on its domain.

2. The action of ι on classical rays in QET is isomorphic to that of the classical ι on classical sets in SET.

Then the operator

$$ R := \iota{}^{\dagger}\iota \tag{6} $$

has spectrum $R = 0, 1, 2, \ldots$ and gives the rank. Dual statements hold for $\kappa = {}^*\iota^* = {}^*\iota$.

Proof. By explicit construction. We note that a classical extensor ψ obeys $\iota\psi = 0$ if and only if ψ has degree $\neq 1$. Such a ψ is an eigenextensor of $R = 0$. We define ι and $\iota\dagger$ by giving matrix elements $|\phi\langle\iota\langle\psi|$ and $|\phi\langle{}^\dagger\iota\langle\psi|$ in a classical basis for QET: They are then defined on all finite linear combinations of classical extensors, which are dense in QET.

The operator ι couples a basic classical extensor ψ of finite degree only to the basic classical extensor $\iota\psi$, and with matrix element 1: For classical ψ and ϕ of finite degree,

$$\begin{aligned} |\phi\langle\iota\langle\psi| &= 1 \qquad \text{if } \phi = \iota\psi \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{7}$$

If either ψ or ϕ has finite codegree, the matrix element (7) vanishes.

The operator ${}^\dagger\iota$ then annihilates any classical extensor ψ that is not of the form $\iota\phi$ with ϕ classical. (In the harmonic oscillator case this singles out the vacuum vector; here there are infinitely many such vectors.) It couples one that is of this form and has rank $R = n$ to ϕ with matrix element n:

$$\begin{aligned} |\phi\langle{}^\dagger\iota\langle\psi| &= n \qquad \text{if } \psi = \iota\phi \quad \text{has rank} \quad R = n \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{8}$$

It is easy to verify that this defines ι and ${}^\dagger\iota$ densely in QET by linearity, and satisfies (5) and (6). ■

The first-degree qets have been supposed to generate all fermion operators; the ι operator is supposed to be the root of spacetime coordinates and momenta.

For sets of the special form $\iota^n 1$, which Peano identified with the integers, the operator $R := \iota^\dagger\iota$ of (6) is precisely the order operator. In general, an eigenextensor of R with eigenvalue $R = n$ has the form $\iota^n\alpha$ where α is not of the form $\iota\beta$. Then n is the order of the eigenextensor if and only if $\alpha = 1$. In general, n is less than or equal to the order. We call the operator R the rank.

The physical need for the brace and ι already arises in quantum mechanics and field quantum field theory when we must couple dynamical variables with spacetime points to define trajectories or fields. As long as spacetime is a classical set, it is possible and customary to use the classical bracket to couple variables with coordinates. If spacetime is a quantum set (which is a Fermi–Dirac ensemble, we recall), then we might use the quantum ι for this purpose, to do quantum field theory on quantum spacetime.

We do not advocate this use of ι. It copies classical field theories too literally. The main field theory is gravity, which describes the causal connection. In the quantum case, it is more natural to use ι to couple spacetime points directly to each other, and so to describe the causal connection by a network rather than a field. With the network as dynamical variable there is no fundamental need for fields. This is the main motivation for Q.

6. QUANTIFIERS

Physicists have dealt with quantification in quantum logics elegantly since quite early in the development of quantum theory. If $\psi \in H$ describes one fermion, so that an ensemble is described by an extensor $\Psi \in E(H)$, then in physics one uses the numerical quantifier $N(\psi)$ (for "the number of fermions of the kind ψ," also called the ψ-occupation number operator) in preference to the Aristotelean quantifiers $\forall$ ("for every fermion") and $\exists$ ("for some fermion"). $N(\psi)$ is a linear operator on $E(H) \to E(H)$ defined in Q as follows.

For any extensor ψ let us write $\psi^{\vee}$ for the linear operator on extensors of left $\vee$-multiplication by ψ; for all $\psi, \chi \in E(H)$,

$$\psi^{\vee}\chi := \psi \vee \chi \in E(H) \tag{9}$$

Similarly we write $\psi^{\wedge}$ for left $\wedge$-multiplication by ψ.

The operator (9) is a creation operator. Its Hermitian adjoint $\dagger\psi^{\vee}\dagger =: {}^{\dagger}(\psi^{\vee})$ is an annihilation operator identical with $(^{*}\psi)^{\wedge}$, left $\wedge$-multiplication by $^{*}\psi$, the Hodge dual of ψ. (We thank G.-C. Rota for pointing this out.)

Let us normalize ψ to 1. Then ψ and $^{\dagger}\psi$ obey "canonical anticommutation relations" and one defines the number operator $N(\psi)$ by

$$N(\psi) := \psi^{\dagger}\psi \tag{10}$$

For every normalized ψ, $N(\psi)$ is a positive linear operator mapping $E(H) \to E(H)$ with eigenvalues, $N' = 0, 1$. The eigenvalue equation

$$N(\psi)\Psi = n\Psi \tag{11}$$

asserts that there are n elements of the kind ψ in Ψ. The eigenvectors of $N(\psi)$ belonging to eigenvalue $n = 0, 1$ are the homogeneous extensors of degree n in ψ. Those with $n = 1$ are of the form $\psi \vee \alpha$ for some α; those with $n = 0$ lack any factor of ψ and are annihilated by $^{\dagger}\psi$. Thus, $N(\psi)$ agrees with the classical notion of the number of ψ's.

The sum over all ψ in a basis b,

$$N := \sum_{\psi \in b} N(\psi) \tag{12}$$

is the total number operator; the eigenvalue equation $N\Psi = n\Psi$ asserts that there are N elements in Ψ. It is easy to construct a universal quantifier $\bigcap$ and existential quantifier $\bigcup$ using these numerical quantifiers.

The classical existential operator $\bigcup: S \to S$ may be defined by the conditions that

$$\bigcup [\{\psi\}] = \psi, \qquad \bigcup [\psi \cup \chi] = [\bigcup \psi] \cup [\bigcup \chi] \tag{13}$$

Similarly, we might wish to define the extensor existential partial operator $\bigvee$: QET→QET (the disjoint union) by the condition that it be linear and for simple extensors obey

$$\bigvee [\{\psi\}] = \psi, \qquad \bigvee [\psi \vee \chi] = \bigvee \psi \vee \bigvee \chi \tag{14}$$

If either ψ or χ has even degree, this leads to the unexpected nonclassical result

$$\bigvee [\{\psi\} \vee \{\chi\}] = \psi \vee \chi = \chi \vee \psi = \bigvee [\{\chi\} \vee \{\psi\}] = -\bigvee [\{\psi\} \vee \{\chi\}] = 0 \tag{15}$$

This relation stems from the difference between the statistics of ψ (quasi-Bose if ψ is of even degree) and $\iota\psi$ (Fermi in every case), or the fact that ι is a superoperator (mixes statistics). We may still imbed classical quantifiers within the quantum theory, however, despite (15). The quantum entity corresponding to an element e of a set is not an extensor ψ, which is a creator, nor a dual extensor ${}^{\dagger}\psi$, which is an annihilator, but the bilinear $\vee$-product $\psi \vee {}^{\dagger}\psi$, which is, according to the next section, a characteristic function, and the unit set of e is represented by $\iota\psi \vee \iota^{\dagger}\psi$. Such even-degree elements commute, and avoid the paradox of (15).

7. FUNCTIONS

While the language of set theory generates an infinite family of nouns with ease and elegance, it stumbles and stutters on verbs. The simplest mapping or function is the arrow, which we write as $\beta \leftarrow \alpha$. It represents an operation that transforms α into β. It creates insuperable problems for set theory. A frequently used expression for $\beta \leftarrow \alpha$ is the set $\{\alpha\} \cup \{\{\alpha\} \cup \{\beta\}\}$ of degree 2. This choice is obviously gratuitous; $\{\alpha\} \cup \{\{\alpha\} \cup \{\beta\}\}$ naturally represents a pair, not a transformation, and could just as well be used to represent the inverse arrow $\alpha \leftarrow \beta$, for example. (For all we know, it is; we have not actually looked the standard convention up, since we will not use it.) Sets do nothing, they simply are. The idea of doing something is unnatural to set theory. A mapping of sets is not naturally a set.

This deficiency in set theory stimulated von Neumann (1925), for example, to provide a variant of set theory which [like the famous λ calculi of Schönfinkel (1924) and Church (1941)] takes the function concept as primitive and defines sets in terms of functions, instead of the converse. One element of the von Neumann set theory survives in von Neumann–Bernays set theory as the distinction between sets and classes, originally a distinction between arguments and functions, which von Neumann made to forestall paradoxes of self-reference. Von Neumann set theory as a whole was too awkward to be practical, however; it preceded the practice of quantum physics, which elegantly constructs functions algebraically, and which we follow in Q.

In quantum field theory we regard a first-degree extensor ψ as a creator, an elemental partial map and its Hilbert dual $\dagger\psi$ as an annihilator, the quasi-inverse partial map. An arrow, an elemental map $A: b \leftarrow a$, is the product of an annihilator of its origin a and a creator of its target b:

$$A = b\dagger a$$

That is, the function A means "Annihilate a and then create b." A more general map is a sum of such arrows. This is how Q represents the throughput operations that connect input and output operations.

It is then possible and natural to express more general mappings, partial mappings, and multivalued mappings as elements of the enlarged extensor algebra $\text{Q}\textsc{et}(\dagger) := \text{Q}\textsc{et} \vee {}^{\dagger}\text{Q}\textsc{et}$ whose extensors are formed from $\downarrow$ by finitely many operations of $\wedge$, $\vee$, ι, κ, $\dagger$, and linear combination, and taking limits. The same construction of functions and more general kinds of mappings may be applied in C set theory by restricting Q($\dagger$) to classical extensors.

It is not supposed that this completes the construction of Q. We lack completely the apparatus of letter variables and their quantification, for example. What we have, however, is enough for the analysis of some simple finite structures, which will occupy us for a time.

8. TIME

We show here the new possibilities opened by Q for a quantum theory of time, leaving the extension to quantum-spacetime to a later paper.

Set theory is not a true language for C physics, but merely a syntax, because the meaning of its symbols varies from application to application. To make physical languages out of set theory it is customary to give various physical interpretations to numbers (and therefore ultimately to the mathematical operation ι used to make numbers) depending on their application, thus introducing non-set-theoretic concepts. For Peano, the operator ι (the classical ι, not the quantum, of course) generated the natural numbers $\mathbb{N}$ as the sequence

$$1, \iota\bullet 1, \iota\iota\bullet 1, \iota\iota\iota\bullet 1, \ldots \tag{16}$$

Since he called $\iota\alpha$ "the successor of α," his $\mathbb{N}$ is at least metaphorically a time axis. In mechanics, however, we use the reals $\mathbb{R}$ as time axis.

In Q we require a uniform interpretation of ι as for the other symbols of Q.

Now ι is the source of the infinity of sets in Q, the principal infinite set in physics is the time axis, and our experience with other infinite sets depends on the infinity of times. Therefore we suppose that ι represents a time step, as suggested by Peano's terminology. The size of this step is a fundamental constant Π with the dimensions of time. We suppose that there is also a quantum of length $c\tau$ associated with Π.

This means that three fundamental constants are built into Q: the quantum of action, the speed of light, and the fundamental time, fixing the scale of nature.

It is not proposed that the construction of Q is completed by this interpretation; that is to be judged by experimental success. To illustrate the use of Q in modeling spacetime, we contrast four time manifolds of time t called quantum, discrete, coherent, and differential.

Quantum Time Manifold. This is the Hilbert space $T < \text{Q}\textsc{et}$ generated by 1 and ι, provided with the successor operator ι as element of structure.

Differential Time Manifold. Here t is a continuous real variable and obeys the commutation relation

$$[\partial, t] = 1 \tag{17}$$

with the time derivative $\partial = \partial/\partial t$, related to energy by $E = ih\partial$. In this standard theory of time, t and ∂ both correspond to diagonalizable unbounded operators.

In the functional (diachronic) version of this theory, which is the one we employ, there is an ideal (unnormalizable) vector $\langle t|$ representing a point of time for each value of the time coordinate t, as well as a dual vector $|t\langle$, with

$$|t'\langle t| = \delta(t' - t) \tag{18}$$

This delta-function becomes a four-dimensional one in spacetime theory and is the source of all the divergencies of field theory. We call the crucial quantity $|t'\langle t|$ the instant form factor.

Such operators of time t and energy $i\partial$ may be represented in Q by

$$\begin{aligned} \partial &= {}^{\dagger}\iota - \iota := \delta, \qquad & t &= \frac{\iota + {}^{\dagger}\iota}{2} := \tau \\ \iota &= t - \tfrac{1}{2}\partial, \qquad & {}^{\dagger}\iota &= t + \tfrac{1}{2}\partial \end{aligned} \tag{19}$$

With this correspondence, and with a positive-definite metric $\|\psi\|$, neither t nor ∂ has normalizable eigenvectors, but both have complete continuous spectra, and in that sense are diagonalizable. Nevertheless, the infinitesimal derivative ∂ is represented exactly by a finite difference operator δ in this ι representation.

Two ways to regularize spacetime are suggested by this theory and exemplified here. One is to replace the continuous spectrum of t by a discrete spectrum and the Dirac δ-function of (18) by a Kronecker δ-function, which is finite. This is the discrete theory of time; it abandons invariance under time translation and the infinite spectrum of energy. The other is to retain the t continuum, but to replace the δ-function of (17) by a regular function that approximates the δ-function but has half-width $\mathbf{n} > 0$. That is, instants of time overlap slightly. This means that t is not a Hermitian operator; an example of this procedure is the coherent theory of time.

Discrete Time Manifold. Now t is not a real variable, but the number operator for units of time or chronons. If we use ${}^{\dagger}\iota$ and ι as bosonic annihilator and creator of the hypothetical chronons, then instead of (19) we have

$$t = \mathbf{n}\iota^{\dagger}\iota = 0, \mathbf{n}, 2\mathbf{n}, \ldots \tag{20}$$

We may retain the usual commutation relation (17) here only if E is a periodic variable, defined only modulo $1/\mathbf{n}$. In this theory the apparent continuity of t is then supposed to be an error due to lack of resolution.

Coherent Time Manifold. In this theory macroscopic spacetime is supposed to be a macroscopic quantum condensation like superconductivity (Finkelstein, 1988). The time variable t of C mechanics and CQ quantum mechanics is supposed to arise from the underlying Q theory as the classical parameter of a coherent state of a boson oscillator. In one toy model of this kind, time and energy are given by

$$\partial := \iota, \qquad t := {}^{\dagger}\iota, \qquad [{}^{\dagger}t, t] = 1 \tag{21}$$

instead of (19), in units of $\mathbf{n} = 1$. With this choice, and a positive-definite metric $\|\psi\|$, the operator t has an overcomplete family of normalized eigenvectors $|t\langle$, and $\partial = {}^{\dagger}t$ has none.

To see this, we imitate the theory of the overcomplete family of "classical" or "coherent" states of the space coordinate x of a harmonic oscillator (Klauder and Sudarshan, 1968), replacing the oscillator coordinate x by the time coordinate t. The role of the harmonic oscillator ground-state vector is played by the vector $\langle 0| = \downarrow$.

We construct an overcomplete set of vectors $\langle t|$ representing instants of time by applying a unitary displacement operator $e^{t\delta}$ to the first flash $\langle 0|$:

$$\langle t| = e^{t\delta} \bullet \langle 0|, \qquad \delta := \iota^{\dagger} - \iota \tag{22}$$

Here the variable t is a real parameter of the coherent instant $\langle t|$. The coherent theory synthesizes the discrete spectrum of the oscillator with the continuous translational symmetry of the time axis. The instant form factor is then the Gaussian

$$|t'\langle t| = \text{ɲ}^{-1} \exp[-\tfrac{1}{2}(t'-t)^2/\text{ɲ}^2] \tag{23}$$

We have restored the fundamental time constant ɲ to show how this form factor approaches a δ-function as $\text{ɲ} \to 0$.

This is a promising regularization of the singularity (19) in the standard theory. It promises, in particular, to eliminate all the infinities of field theory at once.

The next step in this program should be the extension from time to spacetime. What is fundamental is the quantum manifold (QM). It contains the discrete and coherent manifold (ΔM, CM) as different bases, and the differential manifold (dM) as a singular limit of the coherent. The one-dimensional lattice of the above model would then become a four-dimensional one; disclinations in this lattice would be the sources of gravity, and more general defects the sources of other gauge fields, as the dislocations of solid-state physics are the source of the Burgers vector or torsion.

There is a natural way to model the four-dimensional commutation relations

$$[\partial_\mu, x^\nu] = \delta_\mu{}^\nu \tag{24}$$

within Q. One adjoins to QET two generators, basic spinor "fermions," proper first-degree extensors ε_0, ε_1. Pairs of ε's are then pseudoboson vectors. Symmetrized sequences of such pairs provide macroscopic vectors. This construction has been called the "superconducting" vacuum.

The attempt to extend (23) from the time axis to Minkowski spacetime by purely formal analogy fails swiftly. It suggests four annihilators x^μ ($\mu = 0, 1, 2, 3$) and four creators ∂_μ related by

$$\partial_\mu = {}^{\dagger}x_\mu \tag{25}$$

and a point form factor

$$|x'\langle x| = e^{-(x'-x)^{1/2}} \tag{26}$$

where x is a point (x^∞) of Minkowski spacetime and $(x'-x)^2$ is the square of the proper time interval. The form factor (26) is well-behaved for timelike intervals, but diverges for spacelike intervals, violating causality. Moreover, the formal covariant generalization of (21),

$$[{}^{\dagger}x^{\mu}, x^{\nu}] = g^{\mu\nu} \tag{27}$$

combined with the assumption of a normalizable origin $\langle 0|$, with $x^{\mu}\langle 0| = 0$ and $|0\langle 0| = 1$, leads to a nondefinite Hilbert-space metric $\dagger$. This is clearly not the correct extension to spacetime.

A more acceptable extension to Minkowski spacetime is now being sought within the Q framework. Most likely the four-dimensional form factor is not merely an inner product of the form $|x'\langle x|$, but has the form $|x'\langle\sigma\langle x|$ depending on a spacelike surface σ defined by the experimenter. For example, a covariant point form factor in four dimensions of this form which approaches a δ-function as $ᴨ \to 0$ is

$$|x'\langle\sigma\langle x| = \exp[-\tfrac{1}{2}(t'-t)^2/ᴨ^2]\exp[-\tfrac{1}{2}(x'-x)^2/ᴨ^2] \tag{28}$$

where t and x are the components of x^{μ} normal and parallel to σ.

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